TMS165/MSA350 Stochastic Calculus
Written exam Tuesday 4 January 2022 2–6 PM

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Aids: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a $\mathbf{P}$ Brownian motion.

Task 1. Show that if a function $g : [a, b] \to \mathbb{R}$ has finite variation $V_g([a, b]) < \infty$, then g is bounded $\sup_{x\in[a,b]} |g(x)| < \infty$. **(5 points)**

Task 2. Is the process $\{\int_0^t (t-r)^{-\alpha}\, dB(r)\}_{t\geq 0}$ a martingale wrt. the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$?

(5 points)

Task 3. A process $X(t)$ has stochastic differential with diffusion coefficient $\sigma(x) = x\,(1-x)$. Assuming that $X(t) \in (0, 1)$, show that the process $Y(t) = \ln(X(t)/(1-X(t)))$ has a constant diffusion coefficient. **(5 points)**

Task 4. Solve the SDE $dX(t) = B(t)\, dt + X(t)\, dB(t)$ for $t \geq 0$, $X(0) = 1$.

(5 points)

Task 5. Find a change of probability measure $d\mathbf{Q}/d\mathbf{P}$ such that $X(t) = \mathrm{e}^{B(t)}$, $t \in [0, T]$, is a $\mathbf{Q}$ martingale. **(5 points)**

Task 6. Explain how it can be that the rate of convergence for the Euler numerical scheme is quicker for weak solutions than for strong solutions. **(5 points)**

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Solutions to Written Exam 3 January 2022

Task 1. For $x \in [a;b]$ we have $|g(x)-g(a)| \leq V_g([a,x]) \leq V_g([a,b])$ so that $\sup_{x\in[a,b]} |g(x)| \leq \sup_{x\in[a,b]} |g(x)-g(a)| + |g(a)| \leq V_g([a,b]) + |g(a)| < \infty$.

Task 2. No, as $\mathbf{E}\{\int_0^t (t-r)^{-\alpha}\, dB(r) | \mathcal{F}_s^B\} = \int_0^s (t-r)^{-\alpha}\, dB(r) + \mathbf{E}\{\int_s^t (t-r)^{-\alpha}\, dB(r) | \mathcal{F}_s^B\} = \int_0^s (t-r)^{-\alpha}\, dB(r) \neq \int_0^s (s-r)^{-\alpha}\, dB(r)$.

Task 3. Let $f(x) = \ln(x/(1-x))$ so that $f'(x) = 1/(x\,(1-x))$, giving $dY(t) = df(X(t)) = f'(X(t))\, dX(t) + \mu_1(X(t))\, dt = \ldots = dB(t) + \mu_2(X(t))\, dt$.

Task 4. By equation 5.31 in Klebaner's book the solution is $X(t) = \mathrm{e}^{B(t)-t/2}\big(1 + \int_0^t \mathrm{e}^{-B(s)+s/2} B(s)\, ds\big)$.

Task 5. As $d(\mathrm{e}^{B(t)}) = \mathrm{e}^{B(t)}\, dB(t) + \frac{1}{2}\mathrm{e}^{B(t)}\, dt$ we get $\mu_1(t) = \mathrm{e}^{B(t)}/2$, $\sigma_1(t) = \mathrm{e}^{B(t)}$ and $\mu_2(t) = 0$ in equation 10.32 in Klebaner's book giving $d\mathbf{Q}/d\mathbf{P} = \mathrm{e}^{-B(T)/2-T/8}$.

Task 6. See Stig Larsson's lecture notes.